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Gauge independence of the S-matrix in the causal approach

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Abstract. The gauge dependence of the time-ordered products for Yang–Mills theories is analysed in perturbation theory by means of the causal method of Epstein and Glaser together with perturbative gauge invariance. This approach allows a simple inductive proof of the gauge independence of the physical S-matrix.

1. Introduction

The properties of (quantum) gauge invariance and gauge-parameter independence, which are inherent in all kinds of gauge theories, have always been of great interest. In the calculation of physical observables, i.e. S-matrix elements, the question of gauge parameter independence arises automatically. In the usual Lagrangian approach to quantum field theory, the gauge invariance of the classical Lagrangian has to be broken in order to quantize the theory. Therefore, gauge fixing terms which depend on free gauge parameters are added to the Lagrangian. The theory then still has Becchi, Rouet and Stora (BRS) invariance [1]. The gauge parameters drop out in S-matrix elements between physical states. But Green functions *are* gauge dependent in general. On the other hand, it can be shown that Green functions of the special class of gauge invariant operators are independent of the method of gauge fixing and so gauge-parameter independent [2].

In fact, the crucial property of gauge theories which allowed us to show the gauge independence of physical S-matrix elements by path-integral methods is BRS invariance, holding for arbitrary gauge parameters. BRS invariance implies generalized Ward–Takahashi identities first proved by Slavnov and Taylor [3, 4]. One considers then the generating functional $W_\lambda(J)$ of the theory, where λ is a gauge parameter and J the external source coupled to a physical field (e.g. a gauge or quark field). Changing the gauge parameter λ by an infinitesimal amount $d\lambda$ and using the Slavnov–Taylor identities, the desired result can easily be derived [5].

The property of gauge-parameter independence has also recently gained renewed interest in practical problems. For example, the introduction of running couplings can only be achieved by a resummation of certain subsets of Feynman diagrams [6–8] and it is then necessary to define a general procedure for maintaining the gauge independence of the theory. Of course, the significance of such resummed objects is always questionable. Furthermore, the problem has also been discussed in the framework of the background-field model for the electroweak standard model [9].

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It is the aim of this paper to describe the situation from a totally different point of view for the example of pure Yang–Mills theory, without making reference to path-integral methods. Some years ago, some of us [10, 11] began to advocate the causal approach to perturbative quantum field theory, which goes back to a classical paper by Epstein and Glaser [12]. No ultraviolet divergences and only well-defined objects (no interacting fields) appear in this approach. Meanwhile, the method has been applied successfully to full Yang–Mills and massive theories as the electroweak standard model [13].

In the causal approach, the S-matrix is viewed as an operator-valued distribution of the following form:

$$S(g) = \mathbf{1} + \sum_{n=1}^{\infty} \frac{1}{n!} \int \mathrm{d}x_1 \ldots \mathrm{d}x_n \, T_n(x_1, \ldots, x_n) g(x_1) \cdot \ldots g(x_n) \tag{1.1}$$

where $g \in \mathcal{S}$, the Schwartz space of functions of rapid decrease. The T_n are well-defined time-ordered products of the first-order interaction T_1, which specifies the theory. For example, for QCD without matter fields one has

$$T_1(x) = \mathrm{i}g f_{abc} \{ \tfrac{1}{2} : A_{\mu a}(x) A_{\nu b}(x) F_c^{\nu\mu}(x) : - : A_{\mu a}(x) u_b(x) \partial^\mu \tilde{u}_c(x) : \} \tag{1.2}$$

where $F_a^{\nu\mu} = \partial^\nu A_a^\nu - \partial^\mu A_a^\nu$ is the *free* field strength tensor and $u_a, \tilde{u}_a$ are the (fermionic) ghost fields. The asymptotic free fields satisfy the well known commutation relations

$$[A_\mu^{(\pm)}(x), A_\nu^{(\mp)}(y)] = \mathrm{i}g^{\mu\nu} D^{(\mp)}(x-y) + \mathrm{i}\frac{1-\lambda}{\lambda}(\partial_\mu \partial_\nu E)^{(\mp)}(x-y) \tag{1.3}$$

in the so-called λ-gauges, and

$$\{u^{(\pm)}(x), \tilde{u}^{(\mp)}(y)\} = -\mathrm{i}D^{(\mp)}(x-y) \tag{1.4}$$

where D and E will be defined below and all other (anti-)commutators vanish. (For the generalization to the massive case see [14].) The introduction of ghost fields is necessary already at first order to preserve perturbative quantum gauge invariance, which we are going to explain now. It can be written in our case by the help of an appropriately defined gauge charge Q:

$$Q := \lambda \int \mathrm{d}^3x \, \partial_\mu A^\mu(x) \overleftrightarrow{\partial}_0 u(x). \tag{1.5}$$

This leads to the following gauge variations for the fields:

$$[Q, A_\mu] = \mathrm{i}\partial_\mu u \qquad [Q, F_{\mu\nu}] = 0 \qquad \{Q, u\} = 0 \qquad \{Q, \tilde{u}\} = -\mathrm{i}\lambda \partial_\mu A^\mu. \tag{1.6}$$

Obviously, these variations have a simpler structure than those in the BRS case. Perturbative quantum gauge invariance is then expressed by the following condition:

$$[Q, T_n(x_1, \ldots x_n)] = \mathrm{i} \sum_{l=1}^{n} \partial_\mu^{x_l} T_{n/l}^\mu(x_1, \ldots x_n) = (\text{sum of divergences}) \tag{1.7}$$

where $T_{n/l}^\mu$ is a mathematically rigorous version of the time-ordered product

$$T_{n/l}^\mu(x_1, \ldots, x_n) \text{ `=' } T(T_1(x_1) \ldots T_{1/1}^\mu(x_l) \ldots T_1(x_n)) \tag{1.8}$$

$$[Q, T_1(x)] =: \mathrm{i}\partial_\nu T_{1/1}^\nu(x) \tag{1.9}$$

constructed by means of the method of Epstein and Glaser. Note that the usual 4-gluon term is missing in T_1. This term is generated by quantum gauge invariance at second order of perturbation theory.

The paper is organized as follows. In the next section we introduce the asymptotic gauge fields in the covariant λ-gauges and discuss their relation for different λ. Then in the third section we construct a concrete representation in momentum space which is useful for certain computations in the next section. There we discuss the physical subspace and prove gauge independence of the time-ordered products on the physical subspace. This result is a consequence of gauge invariance for any λ. The latter property is investigated in section 5. Two appendices contain the technical details.

2. Asymptotic fields in covariant λ-gauges

We shall use asymptotic gauge fields satisfying the modified wave equation

$$\square A_\mu^{(\lambda)} = (1-\lambda)\partial_\mu \partial^\nu A_\nu^{(\lambda)}. \tag{2.1}$$

Here λ is a real gauge parameter, $\lambda = 1$ corresponds to the Feynman gauge. We have omitted colour indices etc which are unimportant in this section, only the Lorentz structure matters here. The upper index (λ) indicates that the field corresponds to the gauge parameter λ, and we are going to consider the fields with different λ simultaneously.

First we want to solve the Cauchy problem for (2.1) with Cauchy data specified at time $t = 0$ in the whole $\mathbb{R}^3$. For this reason we isolate the highest time derivatives in (2.1)

$$\lambda \partial_0^2 A_0^{(\lambda)} = \triangle A_0^{(\lambda)} + (1-\lambda)\partial_0 \partial^j A_j^{(\lambda)} \tag{2.2}$$

$$\partial_0^2 A_j^{(\lambda)} = \triangle A_j^{(\lambda)} + (1-\lambda)\partial_j (\partial^0 A_0^{(\lambda)} + \partial^l A_l^{(\lambda)}). \tag{2.3}$$

Consequently, in agreement with the ordinary wave equation, the Cauchy data are given by $A_\mu^{(\lambda)}(0, \boldsymbol{x})$ and $(\partial_0 A_\mu^{(\lambda)})(0, \boldsymbol{x})$. Taking the divergence ∂^μ of (2.1) we obtain for $\lambda \neq 0$

$$\square \partial^\mu A_\mu^{(\lambda)} = 0 \tag{2.4}$$

so that $A_\mu^{(\lambda)}$ satisfies the iterated wave equation

$$\square^2 A_\mu^{(\lambda)} = 0. \tag{2.5}$$

The Cauchy problem for this equation is considered in the appendix. The solution can be written in terms of the Lorentz invariant distributions $D(x)$ and $E(x)$:

$$A_\mu^{(\lambda)}(x) = \int_{y_0=0} \mathrm{d}^3 y\, D(x-y) \overset{\leftrightarrow}{\partial}{}_0^y A_\mu^{(\lambda)}(y) + \int_{y_0=0} \mathrm{d}^3 y\, E(x-y) \overset{\leftrightarrow}{\partial}{}_0^y \square A_\mu^{(\lambda)}(y). \tag{2.6}$$

Here, all second- and third-order time derivatives under the last integral must be expressed by spatial derivatives of the Cauchy data by means of (2.2) and (2.3).

It is very important to note that the decomposition (2.6) is Lorentz covariant. Indeed, instead of selecting the plane $y_0 = 0$ we may consider a smooth spacelike surface σ with a surface measure $\mathrm{d}\sigma_\nu(y)$. Then, with the help of Gauss' theorem, the integrals in (2.6) can be written in invariant form

$$\int_{y_0=0} \mathrm{d}^3 y \ldots \overset{\leftrightarrow}{\partial}_0 \rightarrow \int_\sigma \mathrm{d}\sigma_\nu(y) \ldots \overset{\leftrightarrow}{\partial}{}^\nu$$

showing that each term on the r.h.s. of (2.6) is a Lorentz four-vector.

Let us denote the first term in (2.6) which satisfies the ordinary wave equation by $A_\mu^w(x)$ (w stands for wave). The second term denoted by B_μ is equal to

$$B_\mu(x) = (1-\lambda) \int \mathrm{d}^3 y\, E(x-y) \overset{\leftrightarrow}{\partial}_0 \partial_\mu \partial A^{(\lambda)}(y) \tag{2.7}$$

where (2.1) has been inserted. For $\mu = j = 1, 2, 3$ the derivative can be taken out by partial integration, so that

$$B_j(x) = \partial_j \chi(x) \tag{2.8}$$

is a spatial gradient with

$$\chi(x) = (1-\lambda)\int_{y_0=0} \mathrm{d}^3y\, E(x-y)\overleftrightarrow{\partial}_0 \partial A^{(\lambda)}(y). \tag{2.9}$$

However, this is impossible for the zeroth component

$$B_0(x) = \partial_0\chi(x) + B(x). \tag{2.10}$$

The difference $B(x)$ can be transformed as follows

$$\begin{aligned} B(x) = (1-\lambda)\int \mathrm{d}^3y\, &[E(x-y)\overleftrightarrow{\partial}_0 \partial_0^y \partial A^{(\lambda)}(y) - \partial_0^x E(x-y)\overleftrightarrow{\partial}_0 \partial A^{(\lambda)}(y)] \\ &= (1-\lambda)\int \mathrm{d}^3y\, \partial_0^y[E(x-y)\overleftrightarrow{\partial}_0 \partial A^{(\lambda)}(y)] \\ &= (1-\lambda)\int \mathrm{d}^3y\, [\partial_0^y E\partial_0\partial A^{(\lambda)} + E\partial_0^2\partial A^{(\lambda)} - \partial_{0y}^2 E\partial A^{(\lambda)} - \partial_0^y E\partial_0\partial A^{(\lambda)}] \\ &= -(1-\lambda)\int \mathrm{d}^3y\, \Box E(x-y)\partial A^{(\lambda)}(y) \\ &= -(1-\lambda)\int \mathrm{d}^3y\, D(x-y)\partial^\nu A_\nu^{(\lambda)}(y). \end{aligned} \tag{2.11}$$

This shows that the field $B(x)$ also fulfils the wave equation $\Box B = 0$. Therefore, it is tempting to combine it with $A_0^w(x)$. The resulting four-component field

$$A_\mu^L = (A_0^w + B, A_1^w, A_2^w, A_3^w) \tag{2.12}$$

satisfies the wave equation and we have the simple decomposition

$$A_\mu^{(\lambda)} = A_\mu^L + \partial_\mu\chi. \tag{2.13}$$

However, this decomposition has the serious defect of not being covariant (see (2.12)). Therefore, we must make a sharp distinction between the field A_μ^L and the *covariant* field A_μ^F in the Feynman gauge $\lambda = 1$, although both fields satisfy the wave equation and the same commutation relation, as we shall see.

Next we want to quantize the $A^{(\lambda)}$-field. It follows from (2.6) that the commutation relations for arbitrary times must involve the distributions D and E. Then, Poincaré covariance and the singular order $\omega = -2$ of the resulting distribution suggest the following form

$$[A_\mu^{(\lambda)}(x), A_\nu^{(\lambda)}(y)] = \mathrm{i}g_{\mu\nu}D(x-y) + \mathrm{i}\alpha\partial_\mu\partial_\nu E(x-y) \tag{2.14}$$

where a common factor $(h/2\pi)$ has been set =1. When operating with $\Box g^{\kappa\mu} - (1-\lambda)\partial^\kappa\partial^\mu$ on the variable x, we must obtain zero. This determines the parameter α. Using $\Box E(x) = D(x)$, we find

$$[A_\mu^{(\lambda)}(x), A_\nu^{(\lambda)}(y)] = \mathrm{i}g_{\mu\nu}D(x-y) + \mathrm{i}\frac{1-\lambda}{\lambda}\partial_\mu\partial_\nu E(x-y). \tag{2.15}$$

The corresponding commutation relations for the positive- and negative-frequency parts read (see (1.3))

$$[A_\mu^{(\lambda)(-)}(x), A_\nu^{(\lambda)(+)}(y)] = \mathrm{i}g_{\mu\nu}D^{(+)}(x-y) + \mathrm{i}\frac{1-\lambda}{\lambda}(\partial_\mu\partial_\nu E)^{(+)}(x-y). \tag{2.16}$$

Note that the positive-frequency part of the derivative $(\partial_\nu E)$ is well defined, in contrast to $E^{(+)}$.

From the initial values of the D- and E-distributions

$$D(0,\boldsymbol{x})=0 \qquad (\partial_0 D)(0,\boldsymbol{x})=\delta^3(\boldsymbol{x}) \tag{2.17}$$

$$(\partial_0^n E)(0,\boldsymbol{x})=0 \qquad n=0,1,2 \qquad (\partial_0^3 E)(0,\boldsymbol{x})=\delta^3(\boldsymbol{x}) \tag{2.18}$$

we obtain the equal-time commutation relations

$$[\partial_0 A_\mu^{(\lambda)}(x), A_\nu^{(\lambda)}(y)]_0 = \mathrm{i} g_{\mu\nu}\left(1+g_{\mu 0}\frac{1-\lambda}{\lambda}\right)\delta(\boldsymbol{x}-\boldsymbol{y}) \tag{2.19}$$

$$[\partial_0 A_0^{(\lambda)}(x), \partial_0 A_j^{(\lambda)}(y)]_0 = \mathrm{i}\frac{\lambda-1}{\lambda}\partial_j\delta(\boldsymbol{x}-\boldsymbol{y}) \tag{2.20}$$

where the subscript 0 means $x_0 = y_0$. All other commutators are zero. It follows from (2.19) (2.20) that three-dimensional smearing with a space dependent test function $f(\boldsymbol{x})$ is sufficient to obtain a well-defined operator in Fock space.

From the fundamental commutation relations (2.15) the commutators of all other fields can be calculated because they are all expressed by $A_\mu^{(\lambda)}$. We find

$$[\chi(x),\chi(y)]=0 \tag{2.21}$$

$$[A_j^w(x), A_k^w(y)] = \mathrm{i} g_{jk} D(x-y) \tag{2.22}$$

$$[A_0^w(x), A_0^w(y)] = \frac{\mathrm{i}}{\lambda} D(x-y) \tag{2.23}$$

$$[A_\mu^L(x), A_\nu^L(y)] = \mathrm{i} g_{\mu\nu} D(x-y). \tag{2.24}$$

Now, $A_j^w(x)$, $j=1,2,3$ are the spatial components of a covariant vector field satisfying the wave equation. The commutation relations (2.22) are the same as for the Feynman field $A_j^F(x)$. Nevertheless, we cannot identify the two as we shall see in the next section by constructing a concrete representation of the field operators.

3. Concrete representation in momentum space

Most authors who consider the λ-gauges leave the construction of a concrete representation to the reader. We try to be more polite to our readers.

Since three-dimensional smearing is enough to render $A_\mu^{(\lambda)}(x)$ well defined, we will construct all fields as three-dimensional Fourier integrals, leaving aside manifest Lorentz covariance. Our strategy will be to start with a representation of the time-zero fields which satisfies the equal-time commutation relations (2.19), (2.20) and then calculating the time evolution by the formulae of section 2. We follow the somewhat unusual, but mathematically more satisfactory procedure of assuming a Fock space with *positive definite metric* and changing the form of the zeroth component $A_0^{(\lambda)}$ instead [10]. This is very natural in the λ-gauge because the zeroth component plays a special role here, anyway.

We use the usual emission and absorption operators for all four components satisfying

$$[a_\nu^{(\lambda)}(\boldsymbol{p}), a_\mu^{(\lambda)+}(\boldsymbol{q})] = \delta_{\nu\mu}\delta(\boldsymbol{p}-\boldsymbol{q}). \tag{3.1}$$

The adjoint is defined with respect to the positive definite scalar product so that these operators can be represented in the usual way in a Fock space $\mathcal{F}^{(\lambda)}$, if smeared out with test functions $f(\boldsymbol{p}) \in L^2(\mathbb{R}^3)$. In addition we will use the operators for the longitudinal mode

$$a_\parallel^{(\lambda)}(\boldsymbol{p}) = \frac{p^j}{\omega} a_j^{(\lambda)}(\boldsymbol{p}) = -\frac{p_j}{\omega} a_j^{(\lambda)}(\boldsymbol{p}) \tag{3.2}$$

where always $\omega = |\boldsymbol{p}| = p^0$. Introducing the linear combinations

$$\begin{aligned} b_1^{(\lambda)}(\boldsymbol{p}) &= \frac{1}{\sqrt{2}}(a_{\parallel}^{(\lambda)}(\boldsymbol{p}) + a_0^{(\lambda)}(\boldsymbol{p})) \\ b_2^{(\lambda)}(\boldsymbol{p}) &= \frac{1}{\sqrt{2}}(a_{\parallel}^{(\lambda)}(\boldsymbol{p}) - a_0^{(\lambda)}(\boldsymbol{p})) \end{aligned} \tag{3.3}$$

we have the following commutators

$$\begin{aligned} &[b_1^{(\lambda)}(\boldsymbol{p}), b_2^{(\lambda)+}(\boldsymbol{q})] = 0 \\ &[a_\nu^{(\lambda)}(\boldsymbol{p}), b_2^{(\lambda)+}(\boldsymbol{q})] = -\frac{1}{\sqrt{2}}\frac{p_\nu}{\omega}\delta(\boldsymbol{p}-\boldsymbol{q}) \\ &[a_\nu^{(\lambda)+}(\boldsymbol{p}), b_1^{(\lambda)}(\boldsymbol{q})] = -\frac{1}{\sqrt{2}}\frac{p^\nu}{\omega}\delta(\boldsymbol{p}-\boldsymbol{q}). \end{aligned} \tag{3.4}$$

In addition to the adjoint we have to introduce a second conjugation K which appears in all Lorentz covariant expressions and defines the so-called Krein structure [17]. It is defined by

$$a_0^{(\lambda)}(\boldsymbol{p})^K = -a_0^{(\lambda)}(\boldsymbol{p})^+ \qquad a_j^{(\lambda)}(\boldsymbol{p})^K = a_j^{(\lambda)}(\boldsymbol{p})^+ \qquad j = 1, 2, 3. \tag{3.5}$$

Note that $a_\mu^{(\lambda)}, a_\mu^{(\lambda)+}$ are not treated as four-vectors, therefore, we always write the indices below.

The gauge field $A_\mu^{(\lambda)}(x)$ must be self-conjugated $A_\mu^{(\lambda)K} = A_\mu^{(\lambda)}$, in order to get a pseudo-unitary S-matrix. Then, a little experimentation shows that the time-zero fields must be of the following form:

$$\begin{aligned} A_\mu^{(\lambda)}(0,\boldsymbol{x}) = (2\pi)^{-3/2}\int \frac{\mathrm{d}^3p}{\sqrt{2\omega}}\bigg\{& a_\mu^{(\lambda)}(\boldsymbol{p})\mathrm{e}^{\mathrm{i}\boldsymbol{px}} + a_\mu^{(\lambda)K}(\boldsymbol{p})\mathrm{e}^{-\mathrm{i}\boldsymbol{px}} \\ &-\frac{1-\lambda}{2\sqrt{2}\lambda}\bigg[\frac{p_\mu}{\omega}b_1^{(\lambda)}(\boldsymbol{p})\mathrm{e}^{\mathrm{i}\boldsymbol{px}} + \frac{p_\mu}{\omega}b_2^{(\lambda)+}(\boldsymbol{p})\mathrm{e}^{-\mathrm{i}\boldsymbol{px}} \\ &-2g_{\mu 0}b_1^{(\lambda)}(\boldsymbol{p})\mathrm{e}^{\mathrm{i}\boldsymbol{px}} - 2g_{\mu 0}b_2^{(\lambda)+}(\boldsymbol{p})\mathrm{e}^{-\mathrm{i}\boldsymbol{px}}\bigg]\bigg\} \end{aligned} \tag{3.6}$$

$$\begin{aligned} (\partial_0 A_\mu^{(\lambda)})(0,\boldsymbol{x}) = -\mathrm{i}(2\pi)^{-3/2}\int \frac{\mathrm{d}^3p}{\sqrt{2\omega}}\bigg\{& \omega a_\mu^{(\lambda)}(\boldsymbol{p})\mathrm{e}^{\mathrm{i}\boldsymbol{px}} - \omega a_\mu^{(\lambda)K}(\boldsymbol{p})\mathrm{e}^{-\mathrm{i}\boldsymbol{px}} \\ &-\frac{1-\lambda}{2\sqrt{2}\lambda}\bigg[-p_\mu b_1^{(\lambda)}(\boldsymbol{p})\mathrm{e}^{\mathrm{i}\boldsymbol{px}} + p_\mu b_2^{(\lambda)+}(\boldsymbol{p})\mathrm{e}^{-\mathrm{i}\boldsymbol{px}} \\ &-2g_{\mu 0}\omega b_1^{(\lambda)}(\boldsymbol{p})\mathrm{e}^{\mathrm{i}\boldsymbol{px}} + 2g_{\mu 0}\omega b_2^{(\lambda)+}(\boldsymbol{p})\mathrm{e}^{-\mathrm{i}\boldsymbol{px}}\bigg]\bigg\}. \end{aligned} \tag{3.7}$$

It is straightforward to verify the commutation relations (2.15), (2.19) and (2.20).

The fields for arbitrary times can now be found from (2.6). For this purpose we need the following three-dimensional Fourier transforms (for $y_0 = 0$)

$$\int \mathrm{d}^3y\, D(x-y)\mathrm{e}^{\mathrm{i}\boldsymbol{py}} = -\frac{\mathrm{i}}{2\omega}(\mathrm{e}^{\mathrm{i}\omega x^0} - \mathrm{e}^{-\mathrm{i}\omega x^0})\mathrm{e}^{\mathrm{i}\boldsymbol{px}} \tag{3.8}$$

$$\int \mathrm{d}^3y\, E(x-y)\mathrm{e}^{\mathrm{i}\boldsymbol{py}} = \frac{-1}{4\omega^2}\left[\mathrm{e}^{\mathrm{i}\omega x^0}\left(x^0 + \frac{\mathrm{i}}{\omega}\right) + \mathrm{e}^{-\mathrm{i}\omega x^0}\left(x^0 - \frac{\mathrm{i}}{\omega}\right)\right]\mathrm{e}^{\mathrm{i}\boldsymbol{px}}. \tag{3.9}$$

We first compute

$$(\partial^\mu A_\mu^{(\lambda)})(0,\boldsymbol{x}) = \frac{-\mathrm{i}}{\lambda}(2\pi)^{-3/2}\int \mathrm{d}^3p\,\sqrt{\omega}(b_1^{(\lambda)}(\boldsymbol{p})\mathrm{e}^{\mathrm{i}\boldsymbol{px}} - b_2^{(\lambda)+}(\boldsymbol{p})\mathrm{e}^{-\mathrm{i}\boldsymbol{px}}) \tag{3.10}$$

$$(\partial_0\partial^\mu A_\mu^{(\lambda)})(0,\boldsymbol{x}) = \frac{-\sqrt{2}}{\lambda}(2\pi)^{-3/2}\int\frac{\mathrm{d}^3p}{\sqrt{2\omega}}\omega^2(b_1^{(\lambda)}(\boldsymbol{p})\mathrm{e}^{\mathrm{i}\boldsymbol{px}} + b_2^{(\lambda)+}(\boldsymbol{p})\mathrm{e}^{-\mathrm{i}\boldsymbol{px}}). \tag{3.11}$$

Then we obtain from the first term in (2.6)

$$A_j^w(x) = (2\pi)^{-3/2}\int\frac{\mathrm{d}^3p}{\sqrt{2\omega}}(a_j^{(\lambda)}(\boldsymbol{p})\mathrm{e}^{-\mathrm{i}px} + a_j^{(\lambda)+}(\boldsymbol{p})\mathrm{e}^{\mathrm{i}px}) + \partial_j f(x) \tag{3.12}$$

where

$$f(x) = -\mathrm{i}\frac{1-\lambda}{4\lambda(2\pi)^{3/2}}\int\frac{\mathrm{d}^3p}{\omega^{3/2}}[b_1^{(\lambda)}(-\boldsymbol{p})\mathrm{e}^{\mathrm{i}px} - b_2^{(\lambda)+}(-\boldsymbol{p})\mathrm{e}^{-\mathrm{i}px}].$$

The zeroth component behaves differently

$$\begin{aligned}A_0^w(x) = (2\pi)^{-3/2}\int\frac{\mathrm{d}^3p}{\sqrt{2\omega}}\Big[&a_0^{(\lambda)}(\boldsymbol{p})\mathrm{e}^{-\mathrm{i}px} - a_0^{(\lambda)+}(\boldsymbol{p})\mathrm{e}^{\mathrm{i}px}\\ &+\frac{1-\lambda}{\sqrt{2}\lambda}(b_1^{(\lambda)}(\boldsymbol{p})\mathrm{e}^{-\mathrm{i}px} + b_2^{(\lambda)+}(\boldsymbol{p})\mathrm{e}^{\mathrm{i}px})\Big]\\ &-\frac{1-\lambda}{2\sqrt{2}\lambda}(2\pi)^{-3/2}\int\frac{\mathrm{d}^3p}{\sqrt{2\omega}}(b_1^{(\lambda)}(-\boldsymbol{p})\mathrm{e}^{\mathrm{i}px} + b_2^{(\lambda)+}(-\boldsymbol{p})\mathrm{e}^{-\mathrm{i}px}).\end{aligned} \tag{3.13}$$

Next we calculate $\chi(x)$ from (2.9)

$$\begin{aligned}\chi(x) = \frac{1-\lambda}{\sqrt{2}\lambda}(2\pi)^{-3/2}\int\frac{\mathrm{d}^3p}{\sqrt{2\omega}}\Big[&b_1^{(\lambda)}(\boldsymbol{p})\mathrm{e}^{-\mathrm{i}px}\left(x^0 - \frac{\mathrm{i}}{2\omega}\right)\\ &+b_2^{(\lambda)+}(\boldsymbol{p})\mathrm{e}^{\mathrm{i}px}\left(x^0 + \frac{\mathrm{i}}{2\omega}\right)\Big] - f(x)\end{aligned} \tag{3.14}$$

and $B(x)$ from (2.11)

$$\begin{aligned}B(x) = &\frac{\lambda-1}{\sqrt{2}\lambda}(2\pi)^{-3/2}\int\frac{\mathrm{d}^3p}{\sqrt{2\omega}}(b_1^{(\lambda)}(\boldsymbol{p})\mathrm{e}^{-\mathrm{i}px} + b_2^{(\lambda)+}(\boldsymbol{p})\mathrm{e}^{\mathrm{i}px})\\ &-\frac{\lambda-1}{\sqrt{2}\lambda}(2\pi)^{-3/2}\int\frac{\mathrm{d}^3p}{\sqrt{2\omega}}(b_1^{(\lambda)}(-\boldsymbol{p})\mathrm{e}^{\mathrm{i}px} + b_2^{(\lambda)+}(-\boldsymbol{p})\mathrm{e}^{-\mathrm{i}px}).\end{aligned} \tag{3.15}$$

This cancels against the second line in (3.13) so that

$$A_0^{(\lambda)}(x) = (2\pi)^{-3/2}\int\frac{\mathrm{d}^3p}{\sqrt{2\omega}}[a_0^{(\lambda)}(\boldsymbol{p})\mathrm{e}^{-\mathrm{i}px} - a_0^{(\lambda)+}(\boldsymbol{p})\mathrm{e}^{\mathrm{i}px}] + \partial_0(\chi + f). \tag{3.16}$$

The first integral in (3.16) and (3.12) formally agrees with the Feynman field A_μ^F, but the latter is defined by means of different annihilation and creation operators $a_\mu^{(1)}(\boldsymbol{p})$, $a_\mu^{(1)+}(\boldsymbol{p})$

$$A_\mu^F(x) = (2\pi)^{-3/2}\int\frac{\mathrm{d}^3p}{\sqrt{2\omega}}[a_\mu^{(1)}(\boldsymbol{p})\mathrm{e}^{-\mathrm{i}px} + a_\mu^{(1)K}(\boldsymbol{p})\mathrm{e}^{\mathrm{i}px}]. \tag{3.17}$$

In $A_\mu^{(\lambda)}$ the terms with wrong frequencies $\sim b_1^{(\lambda)}(-\boldsymbol{p})$ etc cancel out. Then the resulting decomposition $A_\mu^{(\lambda)} = \tilde{A}_\mu^L + \partial_\mu\tilde{\chi}$ is identical to the one introduced by Lautrup [16].

Until now every field $A^{(\lambda)}$ operates in its own Fock space $\mathcal{F}^{(\lambda)}$, but there must exist a λ-independent intersection of these $\mathcal{F}^{(\lambda)}$ where the gauge-independent objects live. Indeed, in the foregoing equations the λ-dependence is only through the unphysical scalar and longitudinal modes b_1, b_2 (3.3), all equations involving only transverse modes which can be written down contain no λ. Therefore, we can safely identify the transverse emission

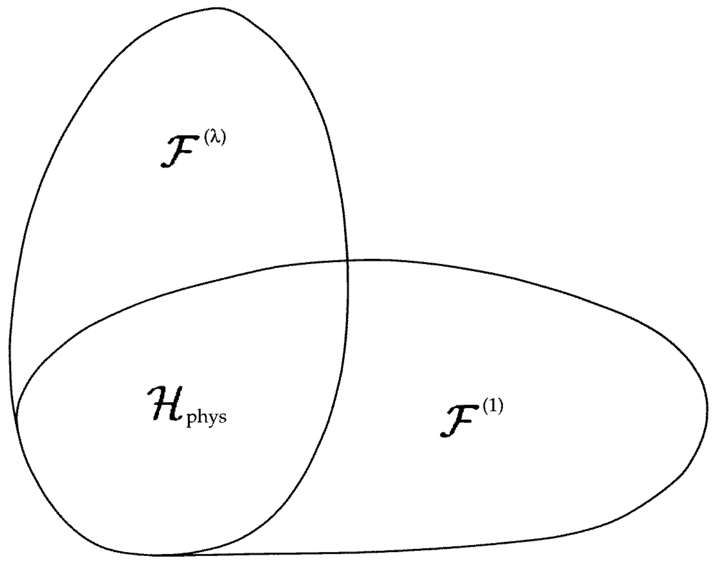

Figure 1. Relation between Fock spaces with different values of the gauge parameter λ.

and absorption operators for different λ. Let $\varepsilon^\mu = (0, \boldsymbol{\varepsilon})$ and $\eta^\mu = (0, \boldsymbol{\eta})$ be two transverse polarization vectors

$$\boldsymbol{p} \cdot \boldsymbol{\varepsilon}(\boldsymbol{p}) = 0 = \boldsymbol{p} \cdot \boldsymbol{\eta}(\boldsymbol{p}) \qquad \boldsymbol{\varepsilon}^2 = 1 = \boldsymbol{\eta}^2 \qquad \boldsymbol{\varepsilon} \cdot \boldsymbol{\eta} = 0. \tag{3.18}$$

Then we put

$$\varepsilon^\mu a_\mu^{(\lambda)}(\boldsymbol{p}) = a_\varepsilon(\boldsymbol{p}) \qquad \eta^\mu a_\mu^{(\lambda)}(\boldsymbol{p}) = a_\eta(\boldsymbol{p}) \tag{3.19}$$

independent of λ. Choosing one unique vacuum Ω for all field operators

$$a_\varepsilon(\boldsymbol{p})\Omega = 0 = a_\eta(\boldsymbol{p})\Omega = b_1^{(\lambda)}(\boldsymbol{p})\Omega = b_2^{(\lambda)}(\boldsymbol{p})\Omega = 0$$

for all $\boldsymbol{p}$ (or rather after smearing with test functions $f(\boldsymbol{p})$), then the different Fock spaces $\mathcal{F}^{(\lambda)}$ hang together (figure 1). Their intersection is the physical subspace $\mathcal{H}_{\text{phys}}$ which is spanned by the transverse states $(a_\varepsilon^+)^m (a_\eta^+)^n \Omega$.

4. Gauge invariance and gauge independence

Now we come to the study of the nilpotent gauge charge Q_λ (1.5)

$$Q_\lambda = \lambda \int \mathrm{d}^3x\, \partial^\mu A_\mu^{(\lambda)}(x) \overleftrightarrow{\partial}_0 u(x) \tag{4.1}$$

where the colour indices are always suppressed if the meaning is clear. The ghost fields u, $\tilde{u}$ are quantized as follows

$$\begin{aligned} &\Box u = 0 \qquad \Box \tilde{u} = 0 \\ &\{u_a(x), \tilde{u}_b(y)\} = -\mathrm{i}\delta_{ab} D(x-y). \end{aligned} \tag{4.2}$$

Since there is no λ-dependence here, they can be represented in the usual way

$$u(x) = (2\pi)^{-3/2} \int \frac{\mathrm{d}^3 p}{\sqrt{2\omega}} (c_2(\boldsymbol{p})\mathrm{e}^{-\mathrm{i}px} + c_1^+(\boldsymbol{p})\mathrm{e}^{\mathrm{i}px}) \tag{4.3}$$

$$\tilde{u}(x) = (2\pi)^{-3/2} \int \frac{\mathrm{d}^3 p}{\sqrt{2\omega}} (-c_1(\boldsymbol{p})\mathrm{e}^{-\mathrm{i}px} + c_2^+(\boldsymbol{p})\mathrm{e}^{\mathrm{i}px}) \tag{4.4}$$

where

$$\{c_i(\boldsymbol{p}), c_j^+(\boldsymbol{q})\} = \delta_{ij}\delta(\boldsymbol{p}-\boldsymbol{q}) \qquad i, j = 1, 2.$$

The conjugation K is extended to the ghost sector by

$$c_2(\boldsymbol{p})^K = c_1(\boldsymbol{p})^+ \qquad c_1(\boldsymbol{p})^K = c_2(\boldsymbol{p})^+ \tag{4.5}$$

so that $u^K = u$ is K self-adjoint and $\tilde{u}^K = -\tilde{u}$. Then Q_λ (4.1), if densely defined, becomes K-symmetric $Q_\lambda \subset Q_\lambda^K$. It is not necessary for the following to give an explicit description of the domain. According to a general result [18], it has a K self-adjoint extension $Q_\lambda^K = Q_\lambda$ which is a closed operator and this is all we need for our purposes.

Using (3.10), (3.11) and (4.3) it is easy to calculate Q_λ in momentum space

$$Q_\lambda = \sqrt{2}\int \mathrm{d}^3p\,\omega(\boldsymbol{p})[b_1(\boldsymbol{p})c_1^+(\boldsymbol{p}) + b_2^+(\boldsymbol{p})c_2(\boldsymbol{p})]. \tag{4.6}$$

For typographical simplicity we have not written the λ-dependence in b_1, b_2. Q_λ together with its adjoint

$$Q_\lambda^+ = \sqrt{2}\int \mathrm{d}^3p\,\omega(\boldsymbol{p})[c_1(\boldsymbol{p})b_1^+(\boldsymbol{p}) + c_2^+(\boldsymbol{p})b_2(\boldsymbol{p})] \tag{4.7}$$

are unbounded closed operators; the unboundedness is not only due to the emission and absorption operators but also because of $\omega(\boldsymbol{p}) = |\boldsymbol{p}|$.

Since Q_λ, Q_λ^+ are closed operators, we have the following direct decompositions of the Fock space

$$\mathcal{F}^{(\lambda)} = \overline{\mathrm{Ran}\,Q_\lambda} \oplus \mathrm{Ker}\,Q_\lambda^+ = \overline{\mathrm{Ran}\,Q_\lambda^+} \oplus \mathrm{Ker}\,Q_\lambda \tag{4.8}$$

where Ran is the range and Ker the kernel of the operator. The overline denotes the closure; note that $\mathrm{Ran}\,Q_\lambda$ is not closed because 0 is in the essential spectrum of Q_λ. Now, $Q_\lambda^2 = 0$ implies $\mathrm{Ran}\,Q_\lambda \perp \mathrm{Ran}\,Q_\lambda^+$, therefore, it follows from (4.8) that

$$\mathcal{F}^{(\lambda)} = \overline{\mathrm{Ran}\,Q_\lambda} \oplus \overline{\mathrm{Ran}\,Q_\lambda^+} \oplus (\mathrm{Ker}\,Q_\lambda \cap \mathrm{Ker}\,Q_\lambda^+). \tag{4.9}$$

The range of Q_λ and Q_λ^+ certainly consists of unphysical states because (4.6) and (4.7) only contains emission operators of unphysical particles (scalar and longitudinal 'gluons' and ghosts). The physical states must therefore be contained in the last subspace in (4.9).

We claim that

$$\mathrm{Ker}\,Q_\lambda \cap \mathrm{Ker}\,Q_\lambda^+ = \mathrm{Ker}\{Q_\lambda, Q_\lambda^+\} \tag{4.10}$$

where the curly bracket is the anticommutator. Indeed, if a vector $f \in \mathcal{F}^{(\lambda)}$ belongs to the l.h.s. that means $Q_\lambda f = 0 = Q_\lambda^+ f$ then it is also contained in the r.h.s. Inversely, if f belongs to the r.h.s. then

$$0 = (f, \{Q_\lambda, Q_\lambda^+\}f) = \|Q_\lambda f\|^2 + \|Q_\lambda^+ f\|^2$$

it is also contained in the l.h.s. Calculating the anticommutator from (4.6), (4.7) we find

$$\{Q_\lambda, Q_\lambda^+\} = 2\int \mathrm{d}^3p\,\omega^2(\boldsymbol{p})[b_1^+(\boldsymbol{p})b_1(\boldsymbol{p}) + b_2^+(\boldsymbol{p})b_2(\boldsymbol{p}) + c_1^+(\boldsymbol{p})c_1(\boldsymbol{p}) + c_2^+(\boldsymbol{p})c_2(\boldsymbol{p})]. \tag{4.11}$$

Up to the (positive) factor ω^2 this is just the particle number operator of the unphysical particles. The physical subspace is characterized by the fact that there are no unphysical particles, hence,

$$\mathcal{H}_{\mathrm{phys}} = \mathrm{Ker}\{Q_\lambda, Q_\lambda^+\} \tag{4.12}$$

and this is a closed subspace. As discussed above, it is the intersection of all $\mathcal{F}^{(\lambda)}$.

We introduce the projection operator P_λ on $\mathcal{H}_{\rm phys}$. It is our aim to prove the gauge independence of the physical S-matrix $P_\lambda S^{(\lambda)}(g)P_\lambda$. The perturbative formulation in terms of time-ordered products $T_n^{(\lambda)}$ (1.1) would be

$$P_\lambda T_n^{(\lambda)} P_\lambda = P_1 T_n^{(1)} P_1 + \text{div}. \tag{4.13}$$

Here div denotes a sum of divergences which vanish after integration with test functions $g(x_1)\ldots g(x_n)$ in the formal adiabatic limit where terms with derivatives of g are neglected. In (4.13) we have compared the physical n-point functions in the λ-gauge with the Feynman gauge $\lambda = 1$.

Gauge independence (4.13) is a direct consequence of gauge invariance (1.7). That (1.7) really holds for arbitrary λ is discussed in the next section. Gauge invariance implies the following important proposition (see [14, equation (5.28)])

$$PT(X_1)PT(X_2)P = PT(X_1)T(X_2)P + \text{div}. \tag{4.14}$$

Here we have omitted indices n and subscripts λ to indicate that (4.14) holds for arbitrary λ and arbitrary n-point functions. For the sake of completeness we give a proof of (4.14) in appendix B.

The proof of gauge independence is by induction on n. The beginning $n = 1$ can be easily verified because $P_\lambda A_\mu^{(\lambda)} P_\lambda = P_1 A_\mu^{(1)} P_1$ and T_1 (1.2) does not depend explicitly on λ. Let us now assume that

$$P_\lambda T_i^{(\lambda)} P_\lambda = P_1 T_i^{(1)} P_1 + \text{div} \tag{4.15}$$

holds for all $i \leqslant n-1$. Then we consider arbitrary products

$$P_\lambda T^{(\lambda)}(X_1) T^{(\lambda)}(X_2) P_\lambda = P_\lambda T^{(\lambda)}(X_1) P_\lambda T^{(\lambda)}(X_2) P_\lambda + \text{div}_1 \tag{4.16}$$

where we have used (4.14). Due to the induction assumption (4.15) this is equal to

$$= P_1 T^{(1)}(X_1) P_1 T^{(1)}(X_2) P_1 + \text{div}_2 = P_1 T^{(1)}(X_1) T^{(1)}(X_2) P_1 + \text{div}_3. \tag{4.17}$$

Here we have used (4.14) again. The causal D-distribution of order n in the Epstein–Glaser construction is a sum of such products (4.16), hence, it follows that

$$P_\lambda D_n^{(\lambda)} P_\lambda = P_1 D_n^{(1)} P_1 + \text{div}. \tag{4.18}$$

All three terms in here have separately causal support, therefore they can individually be split into retarded and advanced parts. The local normalization terms can be chosen in such a way that

$$P_\lambda R_n^{(\lambda)} P_\lambda = P_1 R_n^{(1)} P_1 + \text{div} \tag{4.19}$$

where R denotes the retarded distributions. We must check that this way of normalization is not in conflict with the normalization which we adopt to achieve gauge invariance (see section 5), but this is not the case for the following reason. We decompose

$$T_n = PT_nP + W_n.$$

The condition (4.19) concerns the physical part PT_nP, only. However, the latter is gauge invariant for any normalization

$$QPT_nP - PT_nPQ = 0$$

because $PQ = 0 = QP$. Therefore, the normalization in the proof of gauge invariance involves only the unphysical part W_n. From the gauge independence of the retarded distributions (4.19) we obtain the same result for the n-point distributions

$$P_\lambda T_n^{(\lambda)} P_\lambda = P_1 T_n^{(1)} P_1 + \text{div} \tag{4.20}$$

in the usual way. This completes the inductive proof.

5. Gauge invariance in an arbitrary λ-gauge

Gauge invariance (1.7) has been proven in the Feynman gauge $\lambda = 1$ [11, 15, 19]. Here we summarize and reformulate that proof in a way which is manifestly independent of the choice of λ.

The generator Q_λ (4.1) of the gauge transformations depends on λ, but the parameter λ drops out in the gauge variations of the free fields A^μ, $F^{\mu\nu}$ and u with the exception of $\tilde{u}$ (1.6). However, we shall work with a ghost coupling (1.2) containing the field $\tilde{u}_a$ in the form $\partial_\mu \tilde{u}_a$ only. The gauge variation of the latter field can be written in a λ-independent form

$$\{Q, \partial_\mu \tilde{u}_a\} = -\mathrm{i}\lambda\partial_\mu\partial^\nu A_{a\nu} = \mathrm{i}\partial^\nu F_{a\nu\mu} \tag{5.1}$$

by means of the equation of motion (2.1).

5.1. Gauge invariance at first order

The coupling (1.2) is gauge invariant at first order (1.9) with the 'Q-vertex'

$$T^\nu_{1/1}(x) \stackrel{\text{def}}{=} \mathrm{i}g f_{abc}[: A_{\mu a}(x)u_b(x)F^{\nu\mu}_c(x) : -\tfrac{1}{2} : u_a(x)u_b(x)\partial^\nu\tilde{u}_c(x) :] \tag{5.2}$$

for any value of the gauge parameter λ. The most general coupling which is gauge invariant at first order, symmetrical (Lorentz covariant, $SU(N)$-invariant, P-, T-, C-invariant, pseudo-unitary) and is compatible with renormalizability and contains a non-uniqueness in the ghost sector [21]

$$T_1 + \beta_1\{Q, g f_{abc} : u_a\tilde{u}_b\tilde{u}_c :\} + \beta_2\partial_\mu[\mathrm{i}g f_{abc} : A^\mu_a u_b\tilde{u}_c :] \tag{5.3}$$

$\beta_1, \beta_2 \in \mathbb{R}$ arbitrary, and T_1 is given by (1.2). We shall prove gauge invariance in the case $\beta_1 = 0 = \beta_2$. For general values of $\beta_1, \beta_2 \in \mathbb{R}$ a manifestly λ-independent formulation is impossible, since the fields $\tilde{u}_a$ (without derivative) (1.6) and $\partial_\mu A^\mu_a$ appear in the coupling. (The 'bad' behaviour of $\partial_\mu A^\mu_a$ is explained below in section 5.3) But it has been proven [20] that gauge invariance for $\beta_1 = 0 = \beta_2$ implies gauge invariance for arbitrary $\beta_1, \beta_2 \in \mathbb{R}$ at least at low orders. The argumentation of that proof is of a general kind, such that it applies to any choice of λ.

5.2. Outline of the proof of gauge invariance in the Feynman gauge $\lambda = 1$

In this section we summarize the proof of gauge invariance (1.7) which was given for $\lambda = 1$ in [11, 15, 19]. In section 5.3 we shall see that this proof needs no modifications for arbitrary λ. The proof is by induction on the order n of the perturbation series. The operator gauge invariance (corresponding to (1.7)) of A'_n, R'_n and $D_n = A'_n - R'_n$,

$$[Q, D_n(x_1, \ldots, x_n)] = \mathrm{i}\sum_{l=1}^{n} \partial^{x_l}_\mu D^\mu_{n/l}(x_1, \ldots, x_n) \tag{5.4}$$

has been proven in a straightforward way [11] from the gauge invariance of the T_m, $m \leqslant n-1$. This proof is very instructive because it shows that our definition (1.7) of gauge invariance is adapted to the inductive construction of the T_n's. However, the distribution splitting $D_n = R_n - A_n$ can only be performed in terms of the numerical distributions $d_n = r_n - a_n$. Therefore, we have to express the operator gauge invariance (5.4) by the *Cg-identities* for D_n, the C-number identities for gauge invariance, which imply the operator gauge invariance (5.4).

However, there is a serious problem [11, 19]. Terms with different field operators may compensate, due to identities such as

$$[u(x_1) - u(x_2)]\partial_\mu^{x_1}\delta(x_1 - x_2) + \delta(x_1 - x_2)\partial_\mu u(x_1) = 0. \tag{5.5}$$

Therefore the definition of the C-number distributions in R'_n, A'_n (and therefore in $D_n = R'_n - A'_n$) has a certain ambiguity because terms $\sim (\partial\delta) : A \ldots :$ can mix up with terms $\sim \delta : \partial A \ldots : \sim \delta : F \ldots :$. To get rid of these ambiguities, we choose the convention of *only* applying Wick's theorem (doing nothing else) to

$$A'_n(x_1, \ldots; x_n) = \sum_{Y,Z} \tilde{T}_k(Y) T_{n-k}(Z, x_n) \tag{5.6}$$

where the (already constructed) operator decompositions of $\tilde{T}_k, T_{n-k}$ are inserted. In this way we obtain the so-called *natural* operator decomposition of A'_n

$$A'_n = \sum_{\mathcal{O}} a'_{\mathcal{O}} : \mathcal{O} : \tag{5.7}$$

where the sum runs over all combinations $\mathcal{O}$ of free-field operators. We similarly proceed with $A'_{n/l}$, R'_n and $R'_{n/l}$ and define $d^{(l)}_{\mathcal{O}} \stackrel{\text{def}}{=} r'^{(l)}_{\mathcal{O}} - a'^{(l)}_{\mathcal{O}}$. Then, we split the numerical distributions $d^{(l)}_{\mathcal{O}}$ with respect to their supports into retarded and advanced parts $d^{(l)}_{\mathcal{O}} = r^{(l)}_{\mathcal{O}} - a^{(l)}_{\mathcal{O}}$. Next we define $t'^{(l)}_{\mathcal{O}} \stackrel{\text{def}}{=} r^{(l)}_{\mathcal{O}} - r'^{(l)}_{\mathcal{O}}$ and symmetrize it, which yields $t^{(l)}_{\mathcal{O}}$. The definition

$$T_{n(/l)} \stackrel{\text{def}}{=} \sum_{\mathcal{O}} t^{(l)}_{\mathcal{O}} : \mathcal{O} : \tag{5.8}$$

gives $T_{n(/l)}$ in the *natural* operator decomposition. Note that this procedure fixes the numerical distributions uniquely, up to the normalization in the causal splitting $d^{(l)}_{\mathcal{O}} = r^{(l)}_{\mathcal{O}} - a^{(l)}_{\mathcal{O}}$.

Starting with the natural operator decomposition of D_n (T_n resp.) and $D^\mu_{n/l}$ ($T^\mu_{n/l}$), we commute with Q or take the divergence $\partial_\mu^{x_l}$ according to (5.4) and obtain the *natural operator decomposition* of (5.4) ((1.7) resp.). However, due to (5.5), the Cg-identities for D_n cannot be proven directly by decomposing (5.4). We must go another way: *Instead of proving the operator gauge invariance (1.7), we prove the corresponding Cg-identities (by induction on n), which is a stronger statement*. In this framework the Cg-identities for D_n can be proven by means of the Cg-identities for T_k, $\tilde{T}_k$ at lower orders $1 \leqslant k \leqslant n-1$.

The Cg-identities for T_n are obtained by collecting all terms in the natural operator decomposition of (1.7) which belong to a particular combination $: \mathcal{O} :$ of external field operators. By doing this the arguments of some field operators must be changed by using δ-distributions, i.e. by applying the simple identity

$$: B(x_i)\mathcal{O}(X) : \delta(x_i - x_k) \ldots =: B(x_k)\mathcal{O}(X) : \delta(x_i - x_k) \ldots \tag{5.9}$$

where $X \stackrel{\text{def}}{=} (x_1, x_2, \ldots x_n)$ and $\mathcal{O}(X)$ means the external field operators besides B.

We are now able to give a precise definition of the statement that the Cg-identities hold: *we start with the natural operator decomposition of (1.7). Using several times the identity (5.9), we can obtain an operator decomposition*

$$[Q, T_n(X)] - \mathrm{i}\sum_{l=1}^{n} \partial_l T_{n/l}(X) = \sum_j \tau_j(X) : \mathcal{O}_j(X) : \tag{5.10}$$

(where $\tau_j(X)$ is a numerical distribution and $: \mathcal{O}_j(X) :$ a normally ordered combination of external field operators) which fulfils

$$\tau_j(X) = 0 \qquad \forall j. \tag{5.11}$$

The decomposition (5.10) must be invariant with respect to permutations of the vertices.

A Cg-identity is uniquely characterized by its operator combination $:\mathcal{O}:$. The terms in a Cg-identity are singular of order [11]

$$|\mathcal{O}| + 1 \tag{5.12}$$

at $x = 0$, where

$$|\mathcal{O}| = 4 - b - g_u - g_{\tilde{u}} - d. \tag{5.13}$$

Here, $b, g_u, g_{\tilde{u}}$ are the number of gluons and ghost operators $u, \tilde{u}$, respectively, in $\mathcal{O}$, and d is the number of derivatives on these field operators.

There are no pure vacuum diagrams contributing to (1.7), i.e. terms with no external legs. The disconnected diagrams fulfil the Cg-identities separately. This can be proven easily by means of the Cg-identities for their connected subdiagrams, which hold by the induction hypothesis.

Let us consider a *connected* diagram in the natural operator decomposition of (1.7). We call it *degenerate*, if it has at least one vertex with two external legs; otherwise it is called *non-degenerate*. Let x_i be the degenerate vertex with two external fields, say B_1, B_2. Such a 'degenerate term' has the following form

$$: B_1(x_i)B_2(x_i)B_3(x_{j_1})\ldots B_r(x_{j_{r-2}}) : \Delta(x_i - x_k)t_{n-1}(x_1 - x_n, \ldots \overline{x_i - x_n}, \ldots x_{n-1} - x_n) \tag{5.14}$$

where $k \neq i$, $j_l \neq i$ ($\forall l = 1, \ldots, r-2$) and the coordinate with a bar in t_{n-1} must be omitted. In general, there is a sum of such terms (5.14) belonging to the fixed (degenerate) operator combination $: \mathcal{O} :=: B_1(x_i)B_2(x_i)B_3(x_{j_1})\ldots B_r(x_{j_{r-2}}) :$. For $\Delta(x_i - x_k)$ the following possibilities appear:

(a) $\Delta = D_F,\ \partial D_F,\ \partial_\mu\partial_\nu D_F\ (\mu \neq \nu),\ \partial_\rho\partial_\mu\partial_\nu D_F\ (\mu \neq \nu \neq \rho \neq \mu)$,

(b) $\Delta = \delta^{(4)},\ \partial\delta^{(4)}$.

The $\partial\delta^{(4)}$-terms in (b) cancel [15]. If a degenerate term (5.14) with $\Delta = \delta^{(4)}$ (type (b)) can be transformed in a non-degenerate one by applying (possibly several times) the identity (5.9) only, we call it δ-*degenerate*; if this is not possible we call it *truly degenerate*. All other degenerate terms (i.e. the terms of type (a)) are called truly degenerate, too.

The truly degenerate terms fulfil the Cg-identities separately, by means of the Cg-identities for their subdiagrams [19, section 3.1]. The latter hold by the induction hypothesis. The exception are some tree diagrams at second and third order, which need an explicit calculation [19, section 3.2].

There remain the non-degenerate and δ-degenerate terms, which are linearly dependent. Therefore, *the δ-degenerate terms must be transformed in non-degenerate form by using (5.9)*. In this way we obtain completely *new* Cg-identities, in contrast to the disconnected and the truly degenerate Cg-identities, which rely on Cg-identities at lower orders. Therefore, it is not surprising that the difficult part of the proof of the Cg-identities concerns the non-degenerate $:\mathcal{O}:$ (including δ-degenerate terms). First, one proves the Cg-identities of the non-degenerate and δ-degenerate terms for A'_n, R'_n (and therefore also for D_n) by means of the Cg-identities at lower orders [19, section 4.1]. In the process of distribution splitting the Cg-identities can be violated by local terms only which are singular of order $|\mathcal{O}| + 1$ (5.12), i.e. the possible anomaly has the form

$$a(x_1, \ldots x_n) = \sum_{|b|=0}^{|\mathcal{O}|+1} C_b D^b \delta^{4(n-1)}(x_1 - x_n, \ldots). \tag{5.15}$$

We see that we only have to consider Cg-identities with

$$|\mathcal{O}| \geqslant -1. \tag{5.16}$$

This occurs only for Cg-identities with 2-, 3-, 4-legs and one Cg-identity with 5-legs ($: \mathcal{O} :=: uAAAA :$). For the latter the colour and Lorentz structures exclude an anomaly (5.15) [15]. For the Cg-identities with 2-, 3- and 4-legs we first restrict the constants C_b in the ansatz (5.15) by means of covariance, the SU(N)-invariance and invariance with respect to permutations of the inner vertices. Then we remove the possible anomaly by finite renormalizations of the t-distributions in the Cg-identity. If a certain distribution t appears in several Cg-identities, the different normalizations of t must be compatible. For certain Cg-identities ($: \mathcal{O} :=: uAA :, : uAAA :, : uu\partial\tilde{u}A :$) the removal of the anomaly is only possible, if one uses additional information about the infrared behaviour of the divergences with respect to inner vertices [15].

5.3. The modifications of the proof of gauge invariance for arbitrary λ

Going over to an arbitrary λ-gauge there are two fundamental changes.

(A) The wave equation for the free gauge field A_a^μ is replaced by (2.1). However, in the proof of gauge invariance the equation of motion for A_μ is used in (5.1) only. Therefore, by working always with $\{Q, \partial_\mu \tilde{u}_a\} = \mathrm{i}\partial^\nu F_{a\nu\mu}$ the modification of the equation of motion causes no changes in the proof of gauge invariance.

(B) The commutator $[A_\mu, A_\nu]$ (2.15) has an additional λ-dependent term with the dipole distribution E. Similar changes appear in the positive and negative frequency part of (2.15), as well as in the retarded, advanced and Feynman propagator. All other commutators resp. propagators are independent of λ, e.g. $[A_{a\mu}, F_{b\nu\tau}]$. If we were to work with another ghost coupling $(\beta_1, \beta_2) \neq (0, 0)$ the field $\partial^\mu A_\mu$ would appear, which has a λ-dependent commutator with A_ν

$$[\partial^\mu A_{a\mu}(x), A_{b\nu}(y)] = \frac{\mathrm{i}}{\lambda}\delta_{ab}\partial_\nu D(x-y). \tag{5.17}$$

We now have to check that the explicit form of the AA-commutator resp. propagator is not used in the proof of the Cg-identities.

—Second-order tree diagrams. The explicit form of the propagators is used in the verification of gauge invariance for the second-order tree diagrams, but gauge invariance can only be violated by local terms $\sim (\partial)\delta(x_1 - x_2)$. The latter can only appear if the propagator is of singular order $\omega \geqslant -1$ (see (5.12)), but the AA-propagator (without derivatives) has $\omega = -2$ and, therefore, plays no role in this calculation. In all other propagators (with derivatives) the λ-dependence drops out because the derivatives occur in the antisymmetric F, only. Especially, we conclude that the 4-gluon interaction (which is a normalization term of the second-order tree diagram with external legs $: A(x_1)A(x_1)A(x_2)A(x_2) :$ and is uniquely fixed by gauge invariance [11,19]) is independent of λ and that it is the only local term in $T_2^{(\lambda)}|_{\mathrm{tree}}$.

—δ-degenerate terms. If $\Delta(x_i - x_k)$ in (5.14) originates from an AA-propagator (without derivatives) we know about the singular order $\omega(\Delta) \leqslant \omega([A, A]) + 1 = -1$. Therefore, $\Delta \neq \delta, \partial\delta$ and the set of δ-degenerate terms is unchanged for $\lambda \neq 1$. Of course most t-distributions depend on λ (due to (2.15)), but we conclude that the *Cg-identities belonging to non-degenerate* $: \mathcal{O} :$ (which include the δ-degenerate terms) *are manifestly independent of* λ. (This is obvious for the non-degenerate terms.)

—We turn to the proof of the Cg-identities belonging to non-degenerate $: \mathcal{O} :$ for a' and r' by means of the Cg-identities at lower orders [19, section 4.1]. There one has to show that the operator decomposition of $[Q, A'_n] = [Q, \sum \tilde{T}_k T_{n-k}]$ is unchanged if we interchange the operation $[Q, .]$ with contracting. For this purpose one needs the explicit form of some propagators, but the AA-propagator is not used. The non-trivial step is the cancellation of the terms arising by contracting the commutated leg.

—The same cancellation is used in the proof of the Cg-identities for the truly degenerate terms by means of the Cg-identities for their subdiagrams [19, section 3.1]. Again the explicit form of the AA-propagator plays no role.

We emphasize that (A) and (B) are the only relevant changes for arbitrary λ. Especially the singular order of the numerical distributions (5.12-13) and the symmetries (Lorentz covariance, $SU(N)$-invariance, P-, T- and C-invariance, pseudo-unitarity and invariance with respect to permutations of the vertices) are manifestly independent of λ. Consequently, the ansatz (5.15) for the possible anomalies (in the Cg-identities belonging to non-degenerate $:\mathcal{O}:$) remains the same and the constants C_b in (5.15) can be restricted in the same way. Moreover, the normalization polynomials of the t-distributions are unchanged and, therefore, we can use them to remove the anomalies in the same way. Finally, gauge invariance of third-order tree diagrams, which must be verified explicitly [19, section 3.2] and the proof of the non-trivial 5-legs Cg-identity [15] rely on the $SU(N)$-invariance and Lorentz covariance. Therefore, these parts of the proof also need no change.

Summing up we see that the inductive proof of the Cg-identities is manifestly independent of λ if we choose the ghost coupling $\beta_1 = 0 = \beta_2$ (5.3) and always work with $F_{\mu\nu}$ instead of $\partial_\mu A_\nu$ (5.1).

The coupling to fermionic matter fields (in the fundamental representation) can be added to this model. Gauge invariance holds true if and only if the coupling constants agree (universality of charge). This has been carried out in the Feynman gauge in [20]. There are no changes for arbitrary values of λ.

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Appendix A. Cauchy problem for the iterated wave equation

First we formulate the Cauchy problem for the equation

$$\Box^2 u \equiv (\partial_0^2 - \partial_1^2 - \partial_2^2 - \partial_3^2)^2 u = 0. \tag{A.1}$$

Since (A.1) is of fourth order in time $x_0 = t$, a complete set of Cauchy data at $t = 0$ is given by

$$(\partial_0^n u)(0, \boldsymbol{x}) = u_n(\boldsymbol{x}) \qquad n = 0, 1, 2, 3. \tag{A.2}$$

For simplicity we assume the u_n to be in Schwartz space, then the initial-value problem (A.1), (A.2) has a unique solution. This solution can be constructed by means of the tempered distributions $D(x)$ and $E(x)$, defined by

$$\Box D = 0 \qquad D(0, \boldsymbol{x}) = 0 \qquad (\partial_0 D)(0, \boldsymbol{x}) = \delta^3(\boldsymbol{x}) \tag{A.3}$$

$$\Box^2 E = 0 \qquad (\partial_0^n E)(0, \boldsymbol{x}) = 0 \qquad n = 0, 1, 2 \qquad (\partial_0^3 E)(0, \boldsymbol{x}) = \delta^3(\boldsymbol{x}). \tag{A.4}$$

D is the well known Pauli–Jordan distribution and E is sometimes called dipole distribution and we will compute it.

We now claim that the solution of the Cauchy problem (A.1) and (A.2) is given by

$$u(x) = \int \mathrm{d}^3 y \, [D(x-y)u_1(\boldsymbol{y}) - \partial_0^y D(x-y)u_0(\boldsymbol{y})$$
$$+E(x-y)(u_3 - \triangle u_1)(\boldsymbol{y}) - \partial_0^y E(x-y)(u_2 - \triangle u_0)(\boldsymbol{y})] \tag{A.5}$$

where $\triangle$ denotes the three-dimensional Laplace operator. This formula is the same as the covariant equation (2.6) which is an obvious generalization of the solution of the ordinary wave equation. Using (A.3) and (A.4) it is a simple task to verify (A.1) and (A.2). Therefore it remains for us to construct the dipole distribution E.

From (A.3) and (A.4) we obtain

$$\square E(x) = D(x) \tag{A.6}$$

and we want to obtain E as solution of this equation. We solve this problem in momentum space. The Fourier transform of D is well known

$$\hat{D}(p) = \frac{\mathrm{i}}{2\pi} \, \mathrm{sgn}\, p_0 \delta(p^2) \tag{A.7}$$

so that

$$p^2 \hat{E}(p) = -\frac{\mathrm{i}}{2\pi} \, \mathrm{sgn}\, p_0 \delta(p^2). \tag{A.8}$$

A solution of this equation can immediately be written down by means of the identity

$$p^2 \delta'(p^2) = \frac{\mathrm{d}}{\mathrm{d}p^2} \left(p^2 \delta(p^2)\right) - \delta(p^2) = -\delta(p^2) \tag{A.9}$$

namely

$$\hat{E}(p) = \frac{\mathrm{i}}{2\pi} \, \mathrm{sgn}\, p_0 \delta'(p^2). \tag{A.10}$$

By inverse Fourier transform the initial conditions (A.4) can be verified and $E(x)$ can be computed

$$E(x) = \frac{1}{8\pi} \, \mathrm{sgn}(x_0) \Theta(x^2). \tag{A.11}$$

Note that the positive-frequency part

$$`\hat{E}^{(+)}(p)\text{'} = \frac{\mathrm{i}}{2\pi} \Theta(p_0) \delta'(p^2)$$

is ill defined. This never occurs in rigorous calculations. Only derivatives of E have to be split into positive- and negative-frequency parts (see (1.3)) and these are well defined.

Appendix B

Here we prove the relation (4.14). We start from the orthogonal direct decomposition (4.9) which can be written as

$$\mathbf{1} = P_Q + P_{Q^+} + P \tag{B.1}$$

where P_Q and P_{Q^+} are projection operators onto $\overline{\mathrm{Ran}\, Q}$ and $\overline{\mathrm{Ran}\, Q^+}$ and P projects on $\mathcal{H}_{\mathrm{phys}}$. The operator (4.11)

$$\{Q, Q^+\} \equiv K > 0 \tag{B.2}$$

is positive self-adjoint on the orthogonal complement $\mathcal{H}_{\rm phys}^{\perp}$ of $\mathcal{H}_{\rm phys}$, so that it has an inverse

$$KK^{-1} = P_Q + P_{Q^+} = K^{-1}K \qquad K^{-1}P = 0. \tag{B.3}$$

This allows us to write (B.1) in the form

$$\mathbf{1} = P + QQ^+K^{-1} + Q^+QK^{-1}. \tag{B.4}$$

Now we consider

$$\begin{aligned} PT(X_1)T(X_2)P &= PT(X_1)(P + QQ^+K^{-1} + Q^+QK^{-1})T(X_2)P \\ &= PT(X_1)PT(X_2)P + PT(X_1)QQ^+K^{-1}T(X_2)P \\ &\quad +PT(X_1)Q^+QK^{-1}T(X_2)P. \end{aligned} \tag{B.5}$$

Since $PQ = 0$, the second term is equal to

$$P[T(X_1), Q]Q^+K^{-1}T(X_2)P$$

which is a divergence due to gauge invariance of $T(X_1)$.

In the last term in (B.5) we use the fact that K and, hence, K^{-1} commute with Q which follows easily from the definitions (B.2), (4.11) and (4.6). Then we conclude that

$$PT(X_1)Q^+K^{-1}QT(X_2)P = PT(X_1)Q^+K^{-1}[Q, T(X_2)]P$$

is also a divergence. Consequently,

$$PT(X_1)T(X_2)P = PT(X_1)PT(X_2)P + \text{div}$$

which is the desired relation (4.14).

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